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line through P in the plane α will be at right angles to some line of A and therefore a line of the system. Also any line through P in a plane at right angles to the line AP will be a line of the system. The cone with vertex P has thus degenerated into a pair of planes. Similarly all the points in the plane at infinity are special. For let P be such a point and a that line of A which is at right angles to PA. Then any line through P in the plane of a and PA belongs to the system. Also any line through P in the plane at infinity belongs to the system. For consider any ray a' of A, and let Q be any point on it. Let a plane at right angles to a' at Q meet PA at P'. Then $P'Q$ is a ray of the system, and if Q moves to infinity P' also moves to infinity in the direction of P. The quadratic cone for any point at infinity is thus seen to degenerate into a pair of planes.

In *Special Planes* of the complex the conic touched by the lines of the system degenerates into a pair of points. This is seen to be the case for any plane through A, and also for any plane through the point at infinity in a direction normal to the plane α. These two points differ from other singular points in that *any* line through them is a line of the system.

Several interesting theorems are easily derived from the above discussion. Thus: *Given two flat pencils in different planes, it is possible to find at most four lines passing through any point of space and at right angles to both pencils.* For the two cones determined by the pencils at any point can have at most four elements in common. An unimportant exception arises when the pencils lie in parallel planes, and the point is chosen on the lines joining their centers. *Given two flat pencils in different planes, it is possible to find at most three finite lines lying in any plane and at right angles to both pencils.* For the parabolas determined by the two complexes in any plane can have at most three tangents in common. Taking account of the infinitely distant elements, however, we may state this and the preceding theorem: *The system of lines at right angles to two flat pencils which lie in different planes is of the fourth order and of the fourth class.*

As a final exercise for the student it is not difficult to prove: *The lines at right angles to two given flat pencils and which meet a given line in space are generators of a ruled quartic surface.*

ON THE USE OF PARTIAL DERIVATIVES IN PLOTTING CURVES FROM THEIR EQUATIONS.

By A. M. KENYON, Purdue University.

The article by M. O. Trip in the January, 1914, number of the MONTHLY on "An Application of Partial Derivatives to the Ellipse," has suggested that some further applications which have proved to be practically effective in tracing curves of the second and third degree may be of interest.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a conic section, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ are equations of diameters which bisect all chords parallel to the x- and y-axes respectively, and cut the curve at points where the tangents are horizontal or vertical. If x be eliminated between the equations $f = 0$ and $\partial f/\partial x = 0$ and y between $f = 0$ and $\partial f/\partial y = 0$, the resulting equations determine the horizontal and vertical tangents respectively.

In a central conic all diameters meet in the center; in a parabola all diameters are parallel to the axis of the curve and any two perpendicular diameters meet on the directrix.

Let $\lambda : \mu$ denote the direction of a straight line which makes with the positive x-axis and with the positive y-axis, the angles whose cosines are respectively $\lambda/\sqrt{\lambda^2 + \mu^2}$ and $\mu/\sqrt{\lambda^2 + \mu^2}$. The direction of the line $ax + by + c = 0$ is $-b : a$ or $b : -a$.

If $f = 0$ is a conic, $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a diameter which bisects all chords having the direction $\lambda : \mu$; if the direction $l : m$ of this diameter be determined (in terms of λ and μ), then the equation $\lambda l + \mu m = 0$ determines the directions of the principal axes of the conic; if it is a hyperbola the equation $\mu l - \lambda m = 0$ determines the directions of its asymptotes; if it is a parabola and if $p : q$ is the direction of $\partial f/\partial x = 0$ or $\partial f/\partial y = 0$, then $q(\partial f/\partial x) - p(\partial f/\partial y) = 0$ is the equation of the axis of the curve.

If $f = 0$ is a pair of parallel straight lines, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ represent a line midway between them; if it is a pair of intersecting lines then the equation $\mu l - \lambda m = 0$ determines their directions.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a cubic such that no line in the direction $\lambda : \mu$ cuts it in three points, then $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a conic (parabola, hyperbola, or pair of straight lines) which bisects all chords of the cubic which have this direction. In case this conic is a hyperbola one of its asymptotes has the direction $\lambda : \mu$ and is an asymptote of the cubic; if the conic is a pair of intersecting lines, one of them has the direction $\lambda : \mu$ and the other bisects all chords of the cubic having this direction; if the conic is a pair of parallel lines, both are asymptotes of the cubic.

EXAMPLES. (1) $2x^2 - 2xy + y^2 - 4x + 2y + 1 = 0$.

$\partial f/\partial x = 2(2x - y - 2)$ and $\partial f/\partial y = -2(x - y - 1)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = \sqrt{2}$ and $y = -\sqrt{2}$, horizontal tangents; elimination of y between $f = 0$ and $\partial f/\partial y = 0$ gives $x = 0$ and $x = 2$, vertical tangents. $\partial f/\partial x + \partial f/\partial y = 0$ gives $x = 1$, a diameter bisecting all chords in the direction $1 : 1$; $\partial f/\partial x - \partial f/\partial y = 0$ gives $3x - 2y = 3$, which bisects chords in the direction $1 : -1$.

The direction of $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is $\lambda - \mu : 2\lambda - \mu$ and the equation $\lambda(\lambda - \mu) + \mu(2\lambda - \mu) = 0$ gives $\lambda : \mu = \sqrt{5} - 1 : 2$ and $\sqrt{5} + 1 : -2$, the directions of the principal axes. By drawing chords across these various diameters from the four points of tangency and other points easily found on the curve, a large number of new points may be located.

(2) $$x^2 - 4xy + y^2 + 6x + 24 = 0.$$

$\partial f/\partial x = 2(x - 2y + 3)$ and $\partial f/\partial y = 2(y - 2x)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = -1$, $y = 5$, horizontal tangents;

similarly $x=-2$, $x=4$ are vertical tangents. The equation $\lambda(2\lambda-\mu)+\mu(\lambda-2\mu)=0$ gives $\lambda:\mu=1:1$ and $1:-1$, the directions of the principal axes; the equation $\mu(2\lambda-\mu)-\lambda(\lambda-2\mu)=0$ gives $\lambda:\mu=2+\sqrt{3}:1$ and $2-\sqrt{3}:1$, the directions of the asymptotes.

$$x^2+2xy+y^2-2x-6y+1=0. \tag{3}$$

$\partial f/\partial x=2(x+y-1)$ and $\partial f/\partial y=2(x+y-3)$ have the same direction $1:-1$. $-\partial f/\partial x-\partial f/\partial y=0$ gives $x+y=2$, the axis of the curve. Elimination of x between $f=0$ and $\partial f/\partial x=0$ gives $y=0$, the horizontal tangent. Similarly $x=2$ is the vertical tangent; these tangents cross on the directrix. The focus may now be located.

$$x^3+2y^3-3xy^2-6x-6y=0. \tag{4}$$

$\partial f/\partial x=3(x^2-y^2-2)$ and $\partial f/\partial y=6(y^2-xy-1)$. Lines in the two directions $1:1$ and $2:-1$ cut the cubic in two points only. $\partial f/\partial x+\partial f/\partial y=0$ gives $(x-y+2)(x-y-2)=0$, both asymptotes of the cubic; $2(\partial f/\partial x)-\partial f/\partial y=0$ gives $x^2+xy-2y^2=1$, a hyperbola whose center is at the origin and whose asymptotes are $x-y=0$ and $x+2y=0$, the latter being also an asymptote of the cubic. This hyperbola bisects all chords having the direction $2:-1$.

A METHOD OF SOLVING NUMERICAL EQUATIONS.

By S. A. COREY, Hiteman, Iowa.

The following development of the roots of an equation by Maclaurin's formula applies to both algebraic and transcendental equations, and gives all the roots approximately whenever the conditions involved in the development can be fully complied with.

Let $f(r)=0$ be an equation to be solved, and let a be an approximation to a root r of the equation.

Let us suppose that $f(r)$ is single-valued and analytic in a circle about a as a center and including r. Then

$$f(r)=f(a)+f'(a)(r-a)+\frac{f''(a)}{2!}(r-a)^2+\frac{f'''(a)}{3!}(r-a)^3+\cdots.$$

Hence, by the usual formulas for the reversion of series,[1] putting $z=f(r)-f(a)$, we get

$$\begin{aligned} r-a &= \left(\frac{dr}{dz}\right)_0 z+\left(\frac{d^2r}{dz^2}\right)_0\frac{z^2}{2!}+\left(\frac{d^3r}{dz^3}\right)_0\frac{z^3}{3!}+\cdots\left(\frac{d^nr}{dz^n}\right)_0\frac{z^n}{n!}+\cdots \\ &= A_1^{-1}z-A_2A_1^{-3}\frac{z^2}{2!}+(3A_2{}^2A_1^{-5}-A_3A_1^{-4})\frac{z^3}{3!}+\cdots A_1^{-1}\frac{d}{da}\left(\frac{dr^{n-1}}{dz^{n-1}}\right)_0\frac{z^n}{n!}+\cdots, \end{aligned} \tag{1}$$

where $A_k=d^kf(a)/da^k$ and $A=f(a)$. In practice it is necessary that the devel-

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[1] See, for instance, Goursat, *A Course in Mathematical Analysis*, Vol. 1, §§ 189–190 (first edition).